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PHYS393

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JANUARY EXAMINATIONS 2009

Bachelor of Science: Year 3
Master of Physics: Year 3
Master of Physics: Year 4

STATISTICAL AND LOW TEMPERATURE PHYSICS

MARK SCHEME

TIME ALLOWED: THREE HOURS

INSTRUCTIONS TO CANDIDATES

Answer **all** questions.

Question 1 carries 50% of the total marks.

Questions 2 and 3 each carry 25% of the total marks.

The marks allotted to each part of a question are shown in square brackets.

In the event of a student answering both parts of an either/or question and not clearly crossing out one answer, only the answer to part (a) of the question will be marked.

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1. Answer **all** parts.

a) A set of 6 distinguishable particles can occupy energy states 0, ε, 2ε, 3ε and 4ε. The total energy of the set is 4ε.

(i) Write out the five possible distributions of the particles in the energy states.

The possible distributions are as follows:

0	ε	2ε	3ε	4ε	t
5	0	0	0	1	6
4	1	0	1	0	30
4	0	2	0	0	15
3	2	1	0	0	60
2	4	0	0	0	15
3.33	1.67	0.714	0.238	0.0476	126

Total of 6 particles in each microstate. [1]

Total energy 4ε in each microstate. [1]

(ii) Giving the appropriate formula, or explaining your method, find the number of microstates for each distribution.

Number of microstates in each distribution: see column t in the above table. [1]

Correct formula: $t = \dfrac{N!}{n_1!n_2!n_3!\ldots}$ (or other correct method). [1]

(iii) Evaluate the mean population for each energy state. Give the formula or explain the method that you use.

Mean populations: see bottom row in above table. [1]

Calculated by summing the population of a state in each distribution weighted by the number of microstates for that distribution, and dividing by the total number of microstates (=126). [1]

(iv) If, instead of being distinguishable, the particles had been indistinguishable bosons, evaluate the mean population of each energy state.

For indistinguishable bosons, the microstates are no longer distinct. [1]

Mean populations are (respectively) 3.6, 1.4, 0.6, 0.2, 0.2. [1]

[8]

b) N atoms bound in a solid system can exist in levels of energy ε and 2ε. The level of energy ε contains two states (has a degeneracy of two), while the level of energy 2ε contains a single state.

(i) Write an expression for the partition function Z.

$$Z = \sum_j e^{-\frac{\varepsilon_j}{kT}}$$ [1]

$$\therefore \quad Z = 2e^{-\frac{\varepsilon}{kT}} + e^{-\frac{2\varepsilon}{kT}}$$ [1]

(ii) Using the bridge relation:

$$U = NkT^2 \frac{\partial \ln Z}{\partial T}$$

or otherwise, show that the internal energy U can be written as:

$$U = 2N\varepsilon \frac{1+\exp(-\varepsilon/kT)}{2+\exp(-\varepsilon/kT)},$$

where k is the Boltzmann constant.

$$\frac{\partial \ln Z}{\partial T} = \frac{1}{Z}\frac{\partial Z}{\partial T} = \frac{\frac{\varepsilon}{kT^2}2e^{-\frac{\varepsilon}{kT}} + \frac{2\varepsilon}{kT^2}e^{-\frac{2\varepsilon}{kT}}}{2e^{-\frac{\varepsilon}{kT}} + e^{-\frac{2\varepsilon}{kT}}} = \frac{2\varepsilon}{kT^2}\left(\frac{1+e^{-\frac{\varepsilon}{kT}}}{2+e^{-\frac{\varepsilon}{kT}}}\right)$$ [1]

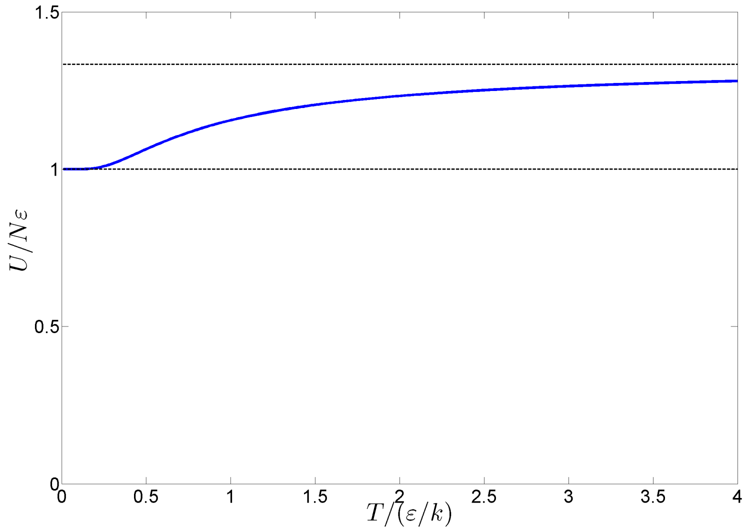
$$\therefore \quad U = NkT^2\frac{2\varepsilon}{kT^2}\left(\frac{1+e^{-\frac{\varepsilon}{kT}}}{2+e^{-\frac{\varepsilon}{kT}}}\right) = 2N\varepsilon\left(\frac{1+e^{-\frac{\varepsilon}{kT}}}{2+e^{-\frac{\varepsilon}{kT}}}\right)$$ [1]

(iii) Derive the limits of U as $T \to 0$ and as $T \to \infty$.

As $T \to 0$, $e^{-\frac{\varepsilon}{kT}} \to 0$, therefore $U \to N\varepsilon$. [1]

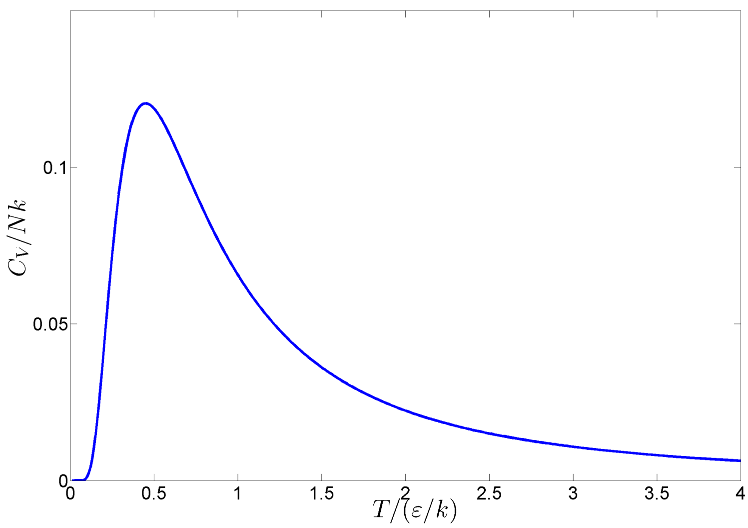
As $T \to \infty$, $e^{-\frac{\varepsilon}{kT}} \to 1$, therefore $U \to \frac{4}{3}N\varepsilon$. [1]

(iv) Sketch the variation of U with T.



Correct qualitative behaviour.	[1]
Zero slope in limit of low temperature.	[1]

(v) Without further differentiation sketch the graph of heat capacity C_V versus T.



Correct qualitative behaviour; peaks at maximum slope of U *vs* T.	[1]
Zero slope in limit of low temperature.	[1]

[10]

c) The Helmholtz free energy F of a system is given by:

$$F = U - TS ,$$

where U is the total internal energy, T the thermodynamic temperature, and S the entropy.

(i) Using the first law of thermodynamics in the form:

$$dU = T\,dS - p\,dV ,$$

show that the entropy is related to the Helmholtz free energy by:

$$S = -\left(\frac{\partial F}{\partial T}\right)_V ,$$

and that the pressure is related to the Helmholtz free energy by:

$$p = -\left(\frac{\partial F}{\partial V}\right)_T .$$

$dF = dU - S\,dT - T\,dS$. [1]

Substituting for $dU = T\,dS - p\,dV$,

we find $dF = -p\,dV - S\,dT$. [1]

At constant volume, $dV = 0$,

$\therefore \quad dF = -S\,dT$, and hence $S = -\left(\frac{\partial F}{\partial T}\right)_V$. [1]

Similarly, at constant temperature, $dT = 0$,

$\therefore \quad dF = -p\,dV$, and hence $p = -\left(\frac{\partial F}{\partial V}\right)_T$. [1]

(ii) In the classical limit, the partition function for a Maxwell-Boltzmann gas can be written as:

$$Z = \frac{V}{V_0}\left(\frac{T}{T_0}\right)^{\frac{3}{2}},$$

where V_0 and T_0 are constants. Using the bridge relation:

$$F = -NkT\ln(Z),$$

find expressions for the entropy and for the pressure of a Maxwell-Boltzmann gas, as functions of volume and temperature.

$$F = -NkT\ln(Z) = -NkT\ln\left(\frac{V}{V_0}\right) - \frac{3}{2}NkT\ln\left(\frac{T}{T_0}\right).$$ [1]

Using $S = -\left(\frac{\partial F}{\partial T}\right)_V$, we have, from differentiating F with respect to T at constant volume, $S = Nk\ln\left(\frac{V}{V_0}\right) + \frac{3}{2}Nk\ln\left(\frac{T}{T_0}\right) + \frac{3}{2}Nk$,

i.e. $S = \frac{3}{2}Nk\left[1 + \frac{2}{3}\ln\left(\frac{V}{V_0}\right) + \ln\left(\frac{T}{T_0}\right)\right]$. [1]

Also, using $p = -\left(\frac{\partial F}{\partial V}\right)_T$, we have, by differentiating F with respect to V at constant temperature, $p = \frac{NkT}{V}$. [1]

[7]

d) A system consists of N fermions that can exist in states uniformly distributed in energy. The number of fermions $n(\varepsilon)d\varepsilon$ in the energy range ε to $\varepsilon+d\varepsilon$ is given by:

$$n(\varepsilon)\,d\varepsilon = \frac{g\,d\varepsilon}{\exp\left(\frac{\varepsilon-\mu}{kT}\right)+1},$$

where $g\,d\varepsilon$ is the number of states in the energy range ε to $\varepsilon+d\varepsilon$, μ is the chemical potential, T is the thermodynamic temperature, and k is Boltzmann's constant.

(i) Show that the chemical potential is given, as a function of temperature, by:

$$\mu(T) = kT\ln\left[\exp\left(\tfrac{N}{gkT}\right)-1\right].$$

We must have: $\int_0^\infty n(\varepsilon)\,d\varepsilon = \int_0^\infty \frac{g\,d\varepsilon}{\exp\left(\frac{\varepsilon-\mu}{kT}\right)+1} = N$.	[1]
Using $\int_0^\infty \frac{dx}{\exp(\beta x-\alpha)+1} = \frac{1}{\beta}\ln[1+\exp(\alpha)]$,	
we find: $\int_0^\infty \frac{g\,d\varepsilon}{\exp\left(\frac{\varepsilon-\mu}{kT}\right)+1} = gkT\ln\left[1+\exp\left(\frac{\mu}{kT}\right)\right] = N$.	[1]
And hence $\mu(T) = kT\ln\left[\exp\left(\tfrac{N}{gkT}\right)-1\right]$.	

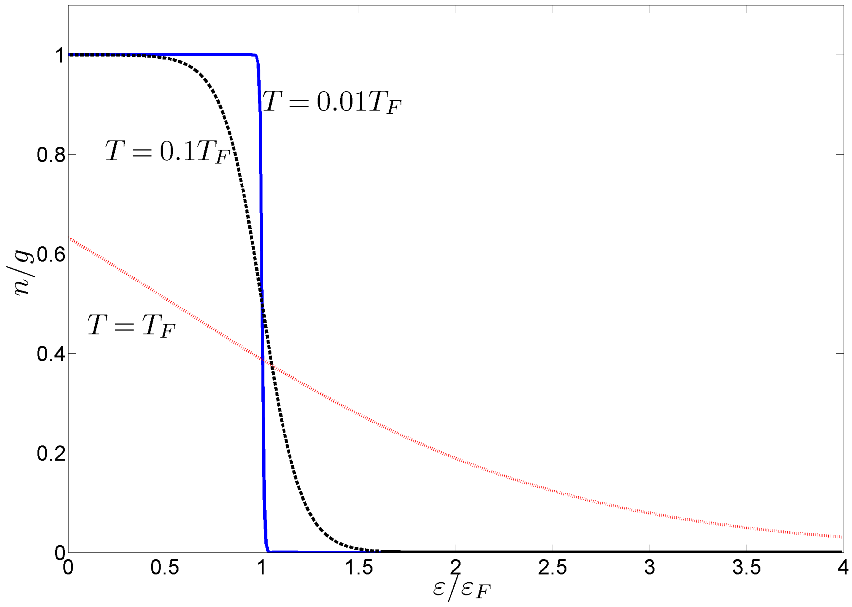
(ii) Show that:

$$\varepsilon_F = \frac{N}{g},$$

where the Fermi energy ε_F is defined as the value of the chemical potential $\mu(T)$ in the limit $T \to 0$. Explain the physical significance of the Fermi energy.

For $T << N/gk$, we have $\exp\left(\tfrac{N}{gkT}\right) >> 1$.	[1]
Hence: $\varepsilon_F = \lim_{T\to 0}\mu(T) = \lim_{T\to 0} kT\ln\left[\exp\left(\tfrac{N}{gkT}\right)\right] = \frac{N}{g}$.	[1]
The Fermi energy is the energy of the highest-energy occupied state in the limit of low temperature.	[1]

(iii) The Fermi temperature T_F is defined by $\varepsilon_F = kT_F$. Sketch a plot showing $n(\varepsilon)$, from $\varepsilon = 0$ to a value $\varepsilon > \varepsilon_F$, in three cases: in the limit $T \to 0$; for $0 < T << T_F$; and for $T > T_F$. Indicate clearly on the horizontal axis on your plot the point $\varepsilon = \varepsilon_F$.



Correct shape ("sharp step") for $T << T_F$ ...	[1]
...with step occurring at $\varepsilon = \varepsilon_F$.	[1]
Correct shape ("smoothed step") for $T < T_F$.	[1]
Correct shape (approximately "exponential decay") for $T > T_F$.	[1]

[9]

[You are given that:

$$\int_0^\infty \frac{dx}{\exp(\beta x - \alpha) + 1} = \frac{1}{\beta}\ln[1 + \exp(\alpha)]. \qquad]$$

e) (i) Explain what is meant by Bose-Einstein condensation.

Below a critical temperature T_B in a boson system,	[1]
the population of the ground state of a series of quantised states becomes very large, and discontinuous with the populations of the excited states.	[1]
The extra population of the ground state is the Bose-Einstein condensation.	[1]

(ii) A system of N atoms of liquid helium-4 (^{4}He) contained in a volume V has a Bose temperature T_B given by:

$$T_B = \left(\frac{h^2}{2\pi mk}\right)\cdot\left(\frac{N}{2.612V}\right)^{\frac{2}{3}},$$

where h is Planck's constant, k is Boltzmann's constant, and m the mass of an atom.

Estimate T_B for liquid ^{4}He, which has a molar volume of 27×10^{-6} m^3.

Since the molar volume is 27×10^{-6} m^3, we have:

$$\frac{N}{V} = \frac{N_A}{27\times10^{-6}\mathrm{m}^{-3}} = 2.23\times10^{28}.$$ [1]

The mass of ^{4}He is approximately four times the proton mass, $\approx 6.68\times10^{-27}$ kg. [1]

Hence we find: $T_B \approx 3.2$ K. [1]

[6]

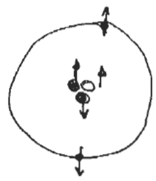
f) i) State the nature of electric current carriers in normal conductors and superconductors

normal conductors: single electrons [1]

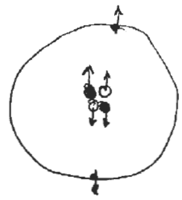
superconductors: electron pairs [1]

[2]

(ii) Draw diagrams showing the constituents of an atom of He^3 one of He^4. Using these, explain why one atom is a fermion and the other is a boson.



2 (1s) electrons spins ↑↓
2 ● protons spins ↑↓
[1] 1 ○ neutron spin ↑



2(1s) electrons spins ↑↓
2 ● protons spins ↑↓
2 ○ neutrons spins ↑↓ [1]

Total atomic angular momentum:

$\vec{F} = \vec{J} + \vec{I}$ [1]

where J is the total electron angular momentum and I is the total nuclear angular momentum. For atoms in the ground state as shown above we have:

He^3 is a fermion because F=1/2 [1]

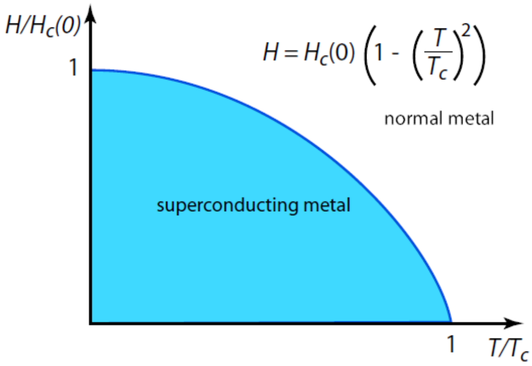
He^3 is a fermion because F=1/2 [1]

[5]

(iii)The critical field $B_C(T)$ in a superconductor at temperature T is related to that at T=0, $B_C(0)$, by the relationship:

$$B_C(T) = B_C(0)\left[1-(T/T_C)^2\right]$$

where T_C is the critical temperature for B=0. Sketch the relation $B_C(T)$ versus T and label the regions of normal conductivity and superconductivity.



[1] shape; [1] regions

[2]

(iv)Lead has T_C=7.2K and $B_C(0)$=0.08T. Is lead at T=6.0K inside a field B=0.04T in a superconducting state?

$$B_C(6) = 0.08\left[1-(6.0/7.2)^2\right] = 0.024T$$

The applied field (0.04T) is higher than the critical field at that temperature (0.024T), so the material is in normal state. [1]

[1]

2. Answer **either** 2(a) **or** 2(b).

a) A Maxwell-Boltzmann gas consists of N monatomic particles in a box with rigid walls. The box is in the shape of a cube with side length L. The wave function for a single particle can be written as:

$$\psi = \psi_0 \sin(k_x x)\sin(k_y y)\sin(k_z z),$$

where ψ_0 is a constant.

(i) Show that possible states have wave numbers k_x, k_y and k_z that satisfy:

$$k_x = \frac{\pi}{L} n_x, \qquad k_y = \frac{\pi}{L} n_y, \qquad k_z = \frac{\pi}{L} n_z,$$

where n_x, n_y and n_z are positive integers (1, 2, 3...) Sketch a plot showing the distribution of states in k-space, and hence show that the number of states with wave number $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$ between k and $k+dk$ is $g(k)\,dk$, given by:

$$g(k)dk = \frac{V}{2\pi^2} k^2 dk,$$

where $V = L^3$ is the volume of the box.

Since the particles cannot penetrate beyond the walls of the box, the wave function must vanish at the walls. [1]

Hence, $\sin(k_x x) = 0$ for $x = 0$ and $x = L$ (and similarly for y and z). [1]

Therefore, $k_x = \pi n_x / L$, for $n_x = 1, 2, 3\ldots$ (and similarly for n_y and n_z).

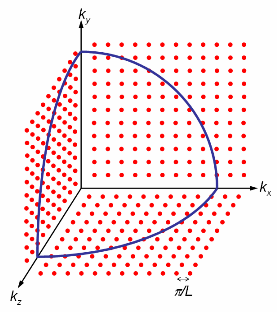


Diagram showing regular grid of points in k-space, for positive values of the components k_x, k_y, k_z, evenly spaced by π/L. [1]

Number of states in range k to $k+dk$ is equal to the number of points in an octant of a spherical shell with inner and outer radii k and $k+dk$.
The volume of this shell is $4\pi k^2 dk/8$. [1]

Number of points per unit volume is $(L/\pi)^3$. Hence $g(k)dk = (4\pi k^2 dk/8)\times(L/\pi)^3$. [1]

[5]

(ii) Write down an expression relating the energy ε of a state to the wave vector k. Hence, show that the number of states with energy between ε and $\varepsilon+d\varepsilon$ is $g(\varepsilon)\,d\varepsilon$, given by:

$$g(\varepsilon)d\varepsilon = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}\sqrt{\varepsilon}\,d\varepsilon\,.$$

$\varepsilon = \dfrac{\hbar^2 k^2}{2m}$. [1]

The number of states between k to $k+dk$ is $g(k)dk = \dfrac{V}{2\pi^2}k^2 dk$.

We have $k^2 = \dfrac{2m\varepsilon}{\hbar^2}$; [1]

therefore, $k = \sqrt{\dfrac{2m\varepsilon}{\hbar^2}}$, and hence $dk = \dfrac{1}{2}\sqrt{\dfrac{2m}{\hbar^2\varepsilon}}d\varepsilon$. [1]

Therefore $\dfrac{V}{2\pi^2}k^2 dk = \dfrac{V}{2\pi^2}\dfrac{2m\varepsilon}{\hbar^2}\times\dfrac{1}{2}\sqrt{\dfrac{2m}{\hbar^2\varepsilon}}d\varepsilon = \dfrac{V}{4\pi^2}\left(\dfrac{2m}{\hbar^2}\right)^{\frac{3}{2}}\sqrt{\varepsilon}d\varepsilon$. [1]

This is the number of states $g(\varepsilon)\,d\varepsilon$ in the energy range ε to $\varepsilon+d\varepsilon$, corresponding to the range of wave vectors k to $k+dk$.

[4]

The number of particles $n(\varepsilon)\,d\varepsilon$ with energy between ε and $\varepsilon+d\varepsilon$ is given by the Boltzmann distribution:

$$n(\varepsilon)\,d\varepsilon = \frac{N}{Z} g(\varepsilon) e^{-\frac{\varepsilon}{k_B T}} d\varepsilon\,,$$

where Z is the partition function, and k_B is Boltzmann's constant.

(iii) Write an expression for Z in terms of the density of states $g(\varepsilon)$. Using this expression for Z, the Boltzmann distribution, and the above expression for the density of energy states $g(\varepsilon)$, show that the number of particles in the Maxwell-Boltzmann gas with energy between ε and $\varepsilon+d\varepsilon$ can be written:

$$n(\varepsilon)\,d\varepsilon = N\frac{2}{\sqrt{\pi}}(k_B T)^{-\frac{3}{2}}\sqrt{\varepsilon}e^{-\frac{\varepsilon}{k_B T}} d\varepsilon\,.$$

$$Z = \int_0^\infty g(\varepsilon) e^{-\frac{\varepsilon}{k_B T}} d\varepsilon\,. \quad [1]$$

Hence, $Z = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \int_0^\infty \sqrt{\varepsilon} e^{-\frac{\varepsilon}{k_B T}} d\varepsilon\,.$ [1]

Using: $\int_0^\infty x^{\frac{1}{2}} e^{-\frac{x}{a}} dx = \frac{\sqrt{\pi}}{2} a^{\frac{3}{2}}$, we find $Z = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \frac{\sqrt{\pi}}{2}(k_B T)^{\frac{3}{2}}\,.$ [1]

The number of particles with energy between ε and $\varepsilon+d\varepsilon$, is:

$$n(\varepsilon)\,d\varepsilon = \frac{N}{Z} g(\varepsilon) e^{-\frac{\varepsilon}{k_B T}} d\varepsilon = \frac{N}{Z}\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \sqrt{\varepsilon} e^{-\frac{\varepsilon}{k_B T}} d\varepsilon\,. \quad [1]$$

Hence, $n(\varepsilon)\,d\varepsilon = N\frac{2}{\sqrt{\pi}}(k_B T)^{-\frac{3}{2}}\sqrt{\varepsilon}e^{-\frac{\varepsilon}{k_B T}} d\varepsilon\,.$

[4]

(iv) Using the above expression for $n(\varepsilon)\,d\varepsilon$, show that the total energy U of the gas is given by:

$$U = \frac{3}{2}Nk_BT\,.$$

The total energy is given by: $U = \int_0^\infty \varepsilon n(\varepsilon)d\varepsilon = N\frac{2}{\sqrt{\pi}}(k_BT)^{-\frac{3}{2}}\int_0^\infty \varepsilon^{\frac{3}{2}}e^{-\frac{\varepsilon}{k_BT}}d\varepsilon$. [1]

Thus, using $\int_0^\infty x^{\frac{3}{2}}e^{-\frac{x}{a}}dx = \frac{3\sqrt{\pi}}{4}a^{\frac{5}{2}}$, we find $U = N\frac{2}{\sqrt{\pi}}(k_BT)^{-\frac{3}{2}}\times\frac{3\sqrt{\pi}}{4}(k_BT)^{\frac{5}{2}}$. [1]

Hence: $U = \frac{3}{2}Nk_BT$.

[2]

(v) Write an expression for the energy ε of a particle in terms of its speed v. Hence, show that the number of particles in the gas with speed between v and $v+dv$ is given by the Maxwell-Boltzmann distribution:

$$n(v)dv = 4\pi N\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}}v^2\exp\left(-\frac{mv^2}{2k_BT}\right)dv\,.$$

$\varepsilon = \frac{1}{2}mv^2$. [1]

The number of particles with energy between ε and $\varepsilon+d\varepsilon$, is:

$$n(\varepsilon)\,d\varepsilon = N\frac{2}{\sqrt{\pi}}(k_BT)^{-\frac{3}{2}}\sqrt{\varepsilon}e^{-\frac{\varepsilon}{k_BT}}d\varepsilon$$ [1]

$\sqrt{\varepsilon} = \sqrt{\frac{m}{2}}v$, and $d\varepsilon = mvdv$. [1]

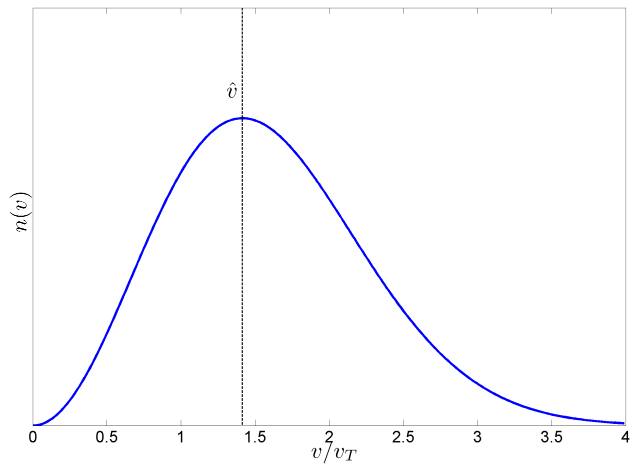
Hence, the number of particles with speed between v and $v+dv$, is:

$$n(v)\,dv = N\frac{2}{\sqrt{\pi}}(k_BT)^{-\frac{3}{2}}\left(\sqrt{\frac{m}{2}}v\right)e^{-\frac{mv^2}{2k_BT}}mvdv,$$ [1]

i.e. $n(v)\,dv = 4\pi N\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}}v^2e^{-\frac{mv^2}{2k_BT}}dv$.

[4]

(vi) Sketch a graph of $n(v)$ versus v, and indicate the most probable speed v.



Correct qualitative behaviour: starting at zero, reaching a peak, then tailing off to zero. [1]

Roughly quadratic dependence on velocity at velocity close to zero. [1]

Most probable speed marked at peak of curve. [1]

[3]

(vii) Find an expression for the mean square speed $\langle v^2 \rangle$, and show that your result is consistent with the above expression for the total energy of the gas.

$$\langle v^2 \rangle = \int_0^\infty v^2 n(v) dv = 4\pi N \left(\frac{m}{2\pi k_B T} \right)^{\frac{3}{2}} \int_0^\infty v^4 e^{-\frac{mv^2}{2k_BT}} dv .$$ [1]

Using: $\int_0^\infty x^4 e^{-\frac{x^2}{b^2}} dx = \frac{3\sqrt{\pi}}{8} b^5$, we find:

$$\langle v^2 \rangle = 4\pi N \left(\frac{m}{2\pi k_B T} \right)^{\frac{3}{2}} \frac{3\sqrt{\pi}}{8} \left(\frac{2k_B T}{m} \right)^{\frac{5}{2}} = 3N \frac{k_B T}{m} .$$ [1]

The total energy of the gas is given by: $U = \frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} N k_B T$. [1]

[3]

[You are given that:

$$\int_0^\infty x^{\frac{1}{2}} e^{-x/a} dx = \frac{\sqrt{\pi}}{2} a^{\frac{3}{2}}, \qquad \int_0^\infty x^{\frac{3}{2}} e^{-x/a} dx = \frac{3\sqrt{\pi}}{4} a^{\frac{5}{2}}, \qquad \int_0^\infty x^4 e^{-x^2/b^2} dx = \frac{3\sqrt{\pi}}{8} b^5 . \text{]}$$

2. (continued)

b) Phonons propagate in a solid shaped as a cube with side length L.

(i) Given that the phonons cannot propagate outside the solid, show that possible states have wave numbers k_x, k_y and k_z that satisfy:

$$k_x = \frac{\pi}{L} n_x, \qquad k_y = \frac{\pi}{L} n_y, \qquad k_z = \frac{\pi}{L} n_z,$$

where n_x, n_y and n_z are positive integers (1, 2, 3...) Sketch a plot showing the distribution of states in k-space, and hence show that the number of states with wave number $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$ between k and $k+dk$ is $g(k)\,dk$, given by:

$$g(k)dk = \frac{3V}{2\pi^2} k^2 dk\,,$$

where $V = L^3$ is the volume of the box.

Since the particles cannot penetrate beyond the walls of the box, the wave function must vanish at the walls. [1]

Hence, $\sin(k_x x) = 0$ for $x = 0$ and $x = L$ (and similarly for y and z). [1]

Therefore, $k_x = \pi n_x / L$, for $n_x = 1, 2, 3\ldots$ (and similarly for n_y and n_z).

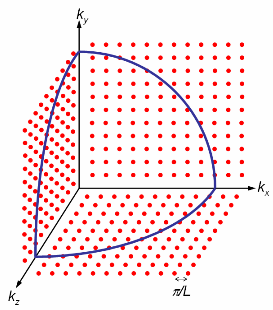


Diagram showing regular grid of points in k-space, for positive values of the components k_x, k_y, k_z, evenly spaced by π/L. [1]

Number of states in range k to $k+dk$ is equal to the three times the number of points (corresponding to three vibrational degrees of freedom, or three phonon polarisations) in an octant of a spherical shell with inner and outer radii k and $k+dk$. The volume of this shell is $4\pi k^2 dk/8$. [1]

Number of points per unit volume is $(L/\pi)^3$. Hence $g(k)dk = (4\pi k^2 dk/8)\times(L/\pi)^3$. [1]

[5]

In the Debye model, the relation between phonon frequency ν and the magnitude k of the wave vector is:

$$\nu = \frac{kc}{2\pi},$$

where c is the velocity of sound.

(ii) Show that the density of phonon states as a function of frequency ν can be written:

$$g(\nu)d\nu = \frac{12\pi V}{c^3}\nu^2 d\nu .$$

We have $k = \frac{2\pi}{c}\nu$, so $dk = \frac{2\pi}{c}d\nu$. [1]

Substituting for k and dk into the expression for $g(k)dk = \frac{3V}{2\pi^2}k^2 dk$, [1]

we find $g(\nu)d\nu = \frac{3V}{2\pi^2}\left(\frac{2\pi}{c}\nu\right)^2 \frac{2\pi}{c}d\nu$, [1]

i.e. $g(\nu)d\nu = \frac{12\pi V}{c^3}\nu^2 d\nu$.

[3]

(iii) A solid containing N atoms supports $3N$ phonons. Show that for such a solid, the cut-off phonon frequency ν_D is given by:

$$\nu_D^3 = \frac{3c^3}{4\pi}\frac{N}{V}.$$

The cut-off frequency ν_D is given by: $\int_0^{\nu_D} g(\nu)d\nu = 3N$. [1]

Using $g(\nu)d\nu = \frac{12\pi V}{c^3}\nu^2 d\nu$, we have $\frac{12\pi V}{c^3}\int_0^{\nu_D}\nu^2 d\nu = 3N$, [1]

and hence $\frac{12\pi V}{c^3}\frac{\nu_D^3}{3} = 3N$, i.e. $\nu_D^3 = \frac{3c^3}{4\pi}\frac{N}{V}$. [1]

[3]

2. b) (continued)

(iv) The phonon energy can be written:

$$U = \int_0^{\nu_D} \frac{12\pi V}{c^3} \nu^2 h\nu \cdot \frac{1}{\exp(h\nu/k_B T) - 1} \cdot d\nu \,.$$

Explain the origin of the factors $h\nu$ and $\dfrac{1}{\exp(h\nu/k_B T) - 1}$.

$h\nu$ is one quantum of phonon energy. [1]

$\dfrac{1}{\exp(h\nu/k_B T) - 1}$ is the probability, in a system of bosons at temperature T, for a state of energy $h\nu$ to be occupied by a particle. [1]

[2]

(v) Writing $y = h\nu/k_B T$, and $y_D = h\nu_D/k_B T$, show that:

$$U = \frac{12\pi V k_B^4 T^4}{c^3 h^3} \int_0^{y_D} \frac{y^3 dy}{e^y - 1} \,.$$

$$U = \frac{12\pi V}{c^3} \int_0^{\nu_D} \cdot \frac{h\nu^3}{\exp(h\nu/k_B T) - 1} \cdot d\nu = \frac{12\pi V}{c^3} \frac{(k_B T)^4}{h^3} \int_0^{\nu_D} \cdot \frac{(h\nu/k_B T)^3}{\exp(h\nu/k_B T) - 1} \cdot \frac{h d\nu}{k_B T}.$$ [1]

Substituting $y = h\nu/k_B T$, and $dy = hd\nu/k_B T$,

and noting that $y = y_D$ when $\nu = \nu_D$, [1]

we find that: $U = \dfrac{12\pi V k_B^4 T^4}{c^3 h^3} \displaystyle\int_0^{y_D} \frac{y^3 dy}{e^y - 1}$

[2]

(vi) Show that, in the low temperature limit $T \to 0$, the total energy U tends to:

$$U \to \frac{4\pi^5 V k_B^4 T^4}{5c^3h^3}.$$

Show also that, in the high temperature limit where $k_B T >> h\nu_D$, the total energy is given approximately by:

$$U \approx 3Nk_B T .$$

In the limit $T \to 0$, we have $y_D = h\nu_D / k_B T \to \infty$, [1]

so $\int_0^{y_D} \frac{y^3 dy}{e^y - 1} \to \int_0^{\infty} \frac{y^3 dy}{e^y - 1} = \frac{\pi^4}{15}$. [1]

Hence $U = \frac{12\pi V k_B^4 T^4}{c^3h^3} \int_0^{y_D} \frac{y^3 dy}{e^y - 1} \to \frac{12\pi V k_B^4 T^4}{c^3h^3} \frac{\pi^4}{15} = \frac{4\pi^5 V k_B^4 T^4}{5c^3h^3}$. [1]

For $k_B T >> h\nu_D$, we can approximate in the integral $e^y - 1 \approx y$,

Hence, $\int_0^{y_D} \frac{y^3 dy}{e^y - 1} \approx \int_0^{y_D} \frac{y^3 dy}{y} = \frac{y_D^3}{3}$, [1]

and so $U = \frac{12\pi V k_B^4 T^4}{c^3h^3} \int_0^{y_D} \frac{y^3 dy}{e^y - 1} \approx \frac{12\pi V k_B^4 T^4}{c^3h^3} \frac{y_D^3}{3} = \frac{4\pi V k_B T}{c^3} \nu_D^3$. [1]

Finally, substituting $\nu_D^3 = \frac{3c^3}{4\pi} \frac{N}{V}$ gives $U = 3Nk_B T$. [1]

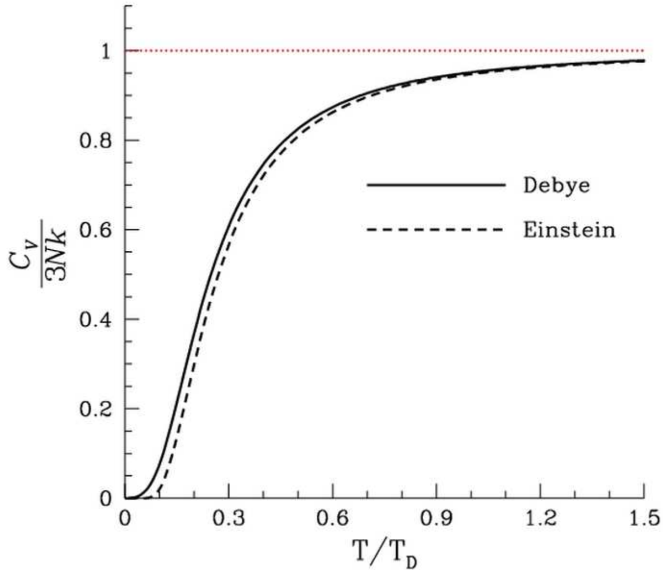
[6]

(vii) Derive expressions for the heat capacity C_V in the low temperature and high temperature limits. Sketch a graph of C_V versus T.

$C_V = \left(\frac{\partial U}{\partial T}\right)_V$. [1]

In the limit $T \to 0$, we have $U \to \frac{4\pi^5 V k_B^4 T^4}{5c^3h^3}$, so $C_V \to \frac{16\pi^5 V k_B^4 T^3}{5c^3h^3}$. [1]

For $k_B T >> h\nu_D$ we have $U = 3Nk_B T$, so $C_V \approx 3Nk_B$. [1]



Correct qualitative behaviour (Einstein curve not required): cubic at low temperature, asymptotic to constant value at high temperature. [1]

[4]

[You are given that:

$$\int_0^\infty \frac{y^3 dy}{e^y - 1} = \frac{\pi^4}{15}. \text{]}$$

3. Answer **__either__** 3(a) **__or__** 3(b).

a) (i) Describe qualitatively the basic features of the theory of superconductivity

Current carriers (electrons in normal conductors) couple in pairs (Cooper pairs) [1]

Electron spins in a pair are anti-alligned giving a zero-spin object; pairs are bosons [1]

Pairs have L=0 and S=0 [1]

Electrons coupling takes place through individual electron coupling to the material lattice [1]

Pairs do no get excited in collisions, hence no resistance [1]

ManyCooper pairs occupy ground state forming Bose-Einstein condensate [1]

[up to 5]

(ii) Explain qualitatively what happens as the temperature of a superconductor rises above the critical temperature

Average energy kT excites electrons breaking up Cooper pairs [2]

[2]

(iii) Explain qualitatively what happens as a magnetic field higher than the critical field is applied to a superconductor

An external field higher than the critical field tends to align electron spins thus breaking up Cooper pairs [2]

[2]

(iv) Describe the isotope effect in superconductivity. What does it prove?

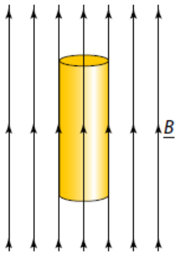
The critical temperature values T_C for different isotopes with atomic mass M of the same element are related through: $T_C \propto \sqrt{1/M}$ [2]

The dependence of T_C on one of the basic lattice parameters proves that Copper pair coupling involves (interactions with) the lattice. [4]

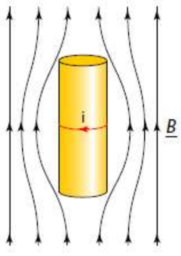
[4]

(v) Describe the Meissner effect

Inside magnetic field B:



normal conductor



superconductor [2]

Material in superconductive state expels all magnetic field lines [1]

Surface currents generate field (M) opposed to the external one (H), leading to zero field inside (B) [1]

$\vec{B} = \mu_0 \left(\vec{M} + \vec{H} \right) = 0 \Rightarrow \vec{H} = -\vec{M}$ perfect diamagnetism [1]

[5]

(vi) What materials exhibit high-T_C superconductivity?

They are ceramic materials [1]

Lightly doped copper-oxide planes [1]

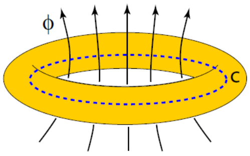
[2]

(vii) Is high-T_C superconductivity understood?

Not at this time – there is no satisfactory theory yet [1]

[1]

(viii) How does the magnetic flux quantum confirm the BCS theory? Mention one important application of magnetic flux quantization.



Superconductor ring placed in magnetic field at $T>T_C$; then temperature lowered below T_C; supercurrents flowing near the surface of the ring maintain flux constant.

BCS theory predicts quantization of magnetic flux in the above configuration:

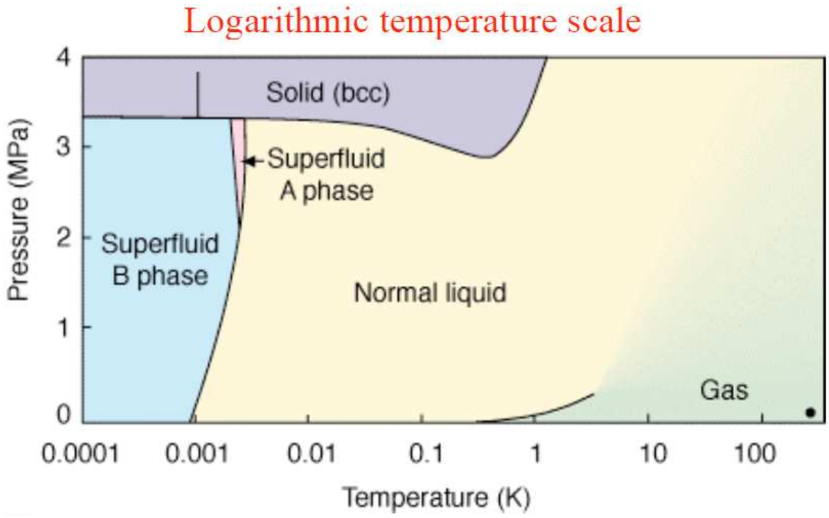
$\Phi = n\phi_0$ with flux quantum: $\phi_0 = \frac{h}{q}$ and q the current carrier charge [2]

Experimental measurements give value of q=2e proving that current in superconductor is carried by electron pairs [2]

[4]

3. (continued)

b) (i) Draw the pressure versus temperature phase diagram for He^3 and label the different phases (assume no external magnetic field).



[1] point for solid-liquid boundary shape

[1] point for normal-superfluid boundary shape

[1] point for drawing A-phase area

[1] point each for noting solid, superfluid, and normal liquid areas

[6]

(ii) Does He^3 become superfluid? If yes, at what temperature? Mention three key features of He^3 behaviour in the relevant temperature range.

He3 becomes superfluid [1]

below 3mK [1]

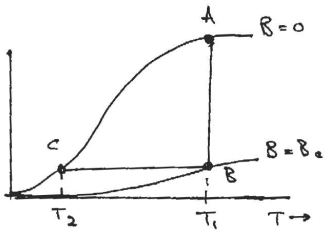
Atoms (fermions) form pairs which are bosons [1]

Atom pairs have S=1, L=1 (magnetic) [1]

Multiple superfluid phases in the presence of magnetic fields; poorly understood [1]

[5]

(iii) Describe nuclear adiabatic demagnetization cooling (outline of technique and principle)



S versus T graph [1]

Example material: ^{63}Cu, nuclear spin=3/2 [1]

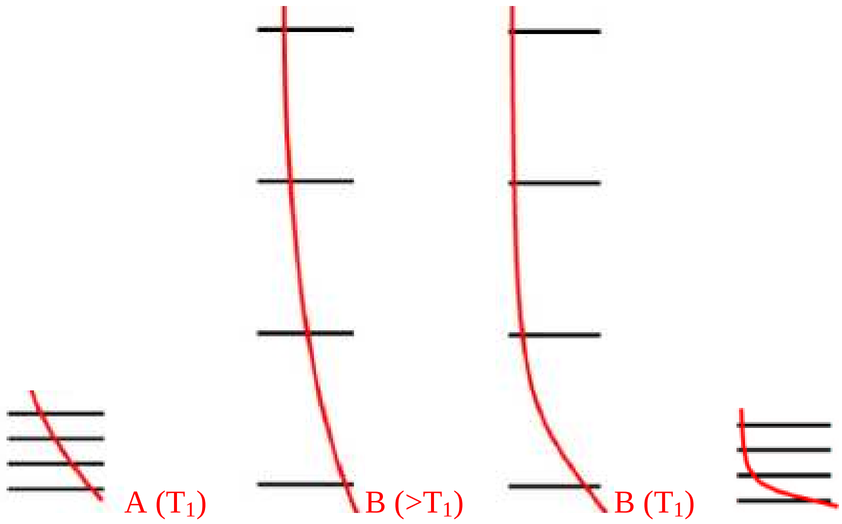
Magnetize sample isothermally at T1 (A to B) [1]

This is achieved with sample connected thermally to heat sink [1]

Isolate thermally the sample [1]

Demagnetize adiabatically (B to C) [1]

Principle:



A (T_1) B (>T_1) B (T_1) C (T_2) [2]

A : almost degenerate states, similar populations [1]

B (>T_1) : large energy gaps, higher total energy (temperature rise) [1]

B (T_1): Large energy gaps, larger population in lower states to keep energy content constant (isothermal magnetization) [1]

C (T_2): lower energy states, population shifted to lower states compared to A, thus lower temperature [1]

[up to 10]

(iv) Describe briefly the two-fluid model for He^4 II and present the main experimental support for it.

Co-existence of superfluid and normal fluid in same volume [1]

At T=0 all liquid is superfluid; at Tλ it is all normal fluid; at intermediate tempratures it is a mixture [1]

Experimental support:

Viscosity measured with small capillary flow close to zero (capillary sees superfluid only) [1]

Viscosity measured with timing of torsional oscillation of disks immersed in liquid is much larger: normal phase liquid sticks to the disks damping oscillation [1]

[4]